Physical consequences of action conservation laws and their applications

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(Received 1 September 2008; published 18 November 2008)

A class of conservation laws containing Hamilton's action integral is introduced for Lagrangian dynamical systems with a single degree of freedom and for the case when the Lagrangian function depends on the second time derivative of the coordinate. The action conservation laws are derived from the invariant properties of the Lagrange-D'Alembert differential variational principle with respect to infinitesimal transformations of the generalized coordinate and time by supposing that the generators of infinitesimal transformations depend on time, a generalized coordinate, and its first and second derivatives with respect to time. These action integral conservation laws are applied to the stability of columns, heat transfer, Thomas-Fermi problems, and other physical phenomena. A direct method for the approximate solution of these problems is combined with the Ritz variational method in order to obtain results of high accuracy.

DOI: 10.1103/PhysRevE.78.056604 PACS number(s): 45.10.Db, 45.20.Jj

I. INTRODUCTION

In this paper we introduce a special class of conservation laws of Lagrangian dynamical systems with a single degree of freedom, containing Hamilton's action integral. The existence of this class of conservation laws was first noted in [1] (pp. 144–149) in the form

$$\psi(t,x,\dot{x}) + \int L(t,x,\dot{x})dt = C = \text{const}, \tag{1}$$

where $L(t,x,\dot{x})$ is the given Lagrangian function of the dynamical system, t is the time, x is the generalized coordinate, and $\dot{x}=dx/dt$ is the generalized velocity. The integral in (1) is usually termed Hamilton's action integral. It is assumed that by differentiating (1) with respect to time and using the differential equation of motion in the form of the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0, \tag{2}$$

expression (1) becomes identical to Eq. (2).

The paper is aimed to the improvement of direct variational methods for obtaining approximate solutions of various Lagrangian systems.

Example A. As a simple example consider the harmonic oscillator $\ddot{x}+\lambda x=0$, whose Lagrangian function is of the form $L=\frac{1}{2}\dot{x}^2-\frac{\lambda}{2}x^2$. The action conservation law is found to be

$$\frac{1}{2}x\dot{x} - \int\left(\frac{1}{2}\dot{x}^2 - \frac{\lambda}{2}x^2\right)dt = C = \text{const}. \tag{3}$$

To demonstrate the utility of this conservation law, we find the first eigenvalue of the following boundary-value problem ([2], pp. 179–180): $\ddot{x}+\lambda x=0$, $x(0)=x(1)+\dot{x}(1)=0$. As suggested in [2], we select an approximate polynomial for x as $x(t)=2035t-1400t^3+294t^5-36t^7+6t^8$, which identically satisfies the boundary conditions. Substituting this into (3) we find from $C(t=0)=C(t=1)$ that $\lambda_{\text{approx}}=\frac{45\,081\,748\,707}{10\,953\,180\,413}=4.115\,859$. The exact solution is the first non-negative root of the equation $\tan\sqrt{\lambda}+\sqrt{\lambda}=0$; i.e., numerical solution of this equation $\lambda_{\text{exact}}=4.115\,858$ agrees with λ_{approx} up to the seventh significant digit.

In this study we also consider action conservation laws of the form

$$\theta(t,x,\dot{x},\ddot{x},\dddot{x}) + \int L(t,x,\dot{x},\ddot{x})dt = D = \text{const} \tag{4}$$

for the Lagrangian $L(t,x,\dot{x},\ddot{x})$, where $\ddot{x}=d^2x/dt^2$ and whose Euler-Lagrange equation is

$$\frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{x}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \frac{\partial L}{\partial x} = 0. \tag{5}$$

Example B. Consider the fourth-order boundary-value problem

$$\ddddot{x} + \lambda\ddot{x} + 20x = 0, \quad \text{with } x(\pm 1) = \ddot{x}(\pm 1) = 0, \tag{6}$$

whose Lagrangian function is

$$L = \frac{1}{2}\ddot{x}^2 - \frac{\lambda}{2}\dot{x}^2 + 10x^2. \tag{7}$$

The action conservation law is of the form

$$-\frac{1}{2}(\lambda x + \ddot{x})x + \frac{1}{2}\dot{x}\ddot{x} - \int\left(\frac{1}{2}\ddot{x}^2 - \frac{\lambda}{2}\dot{x}^2 + 10x^2\right)dt = D = \text{const}. \tag{8}$$

The problem (6) represents an inextensible elastic rod pinned at both ends and loaded by two axial concentrated forces. The lateral displacement of the rod is resisted by an elastic distributed force of the Winkler type (for more details see [6], p. 140). To find the smallest eigenvalue λ for this prob-

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lem, we turn again to an approximate polynomial suggested in [2] p. 218, $x=61-75t^2+15t^4-t^6$, which satisfies identically the boundary conditions in (6). Substituting this into (8) and integrating, we find that $D(t=-1)=D(t=+1)$ and that $\lambda=10.573\,1$, which is identical to the exact solution obtained by means of a different method ([2], p. 219).

It is of importance to stress that together with the action conservation law which is usually defined in a given time interval $[t_0,t_1]$ we will also use a direct method based on Hamilton's variational principle, which states that among all varied paths (i.e., admissible trajectories) connecting two given configurations $x(t_0)=A=\text{const}$ and $x(t_1)=B=\text{const}$, for the given time interval, the actual motion makes the action integral I stationary: namely,

$$\delta I = \delta \int_{t_0}^{t_1} L\, dt = 0. \tag{9}$$

For many important practical cases, the Hamilton action integral I is *minimal* along the actual motion.

Note that the quantity $\int L\, dt$ is named in several (nonuniversal [3], p. 35) ways in the literature [4,5]. We call $\int L\, dt$ the action as in Goldstein [4], p. 36, and Santilli [3], p. 35, while Leech [5], p. 58, calls this quantity the Hamilton principal function.

In what follows we shall demonstrate that by combining, at the same time, action conservation laws with Hamilton's variational principle, many important linear and nonlinear and also rheonomic and scleronomic two-point boundary-value problems can be successfully solved.

It will be demonstrated in the following section that finding the functions $\psi(t,x,\dot{x})$ and $\theta(t,x,\dot{x},\ddot{x},x)$ figuring in (1) and (4) is intimately connected with Noether's theory of finding the conservation laws of Lagrangian dynamical systems.

II. TRANSFORMATION PROPERTIES OF CENTRAL LAGRANGIAN EQUATIONS

A. Case $L=L(t,x,\dot{x})$

In this section we first describe the forms of infinitesimal transformations used here. The symbol δ will denote a simultaneous or Lagrangian variation: a representative point A which is on the actual path $x(t)$ is correlated to an infinitesimally close point B occupied at the same instant of time on the varied path $\bar{x}(t)$ by the relation

$$\bar{x}(t) = x(t) + \delta x. \tag{10}$$

A very important property of simultaneous variations is that the symbol of variation, δ, is commutative with the symbol of differentiation with respect to time: namely,

$$\delta\left(\frac{d^k}{dt^k}\right) - \left(\frac{d^k}{dt^k}\right)\delta = 0, \quad k=1,2,3,\ \dots\ . \tag{11}$$

At the same time, let us enlarge the class of infinitesimal variations by taking into account an infinitesimal deformation of time:

$$\bar{t} = t + \Delta t. \tag{12}$$

We define the generalized (or nonsimultaneous) variation Δx of the generalized coordinate as

$$\Delta x = \delta x + \dot{x}\,\Delta t, \quad \text{i.e.,}\ \ \delta x = \Delta x - \dot{x}\,\Delta t. \tag{13}$$

By taking the kth time derivative of this expression and using (11), we have

$$\delta\left(\frac{d^k x}{dt^k}\right) = \frac{d^k}{dt^k}(\delta x) = \frac{d^k}{dt^k}(\Delta x - \dot{x}\,\Delta t), \quad k=1,2,3,\ \dots\ . \tag{14}$$

In what follows we shall suppose that the nonsimultaneous variations Δx and Δt depend on the time t, generalized coordinate x, generalized velocity $\dot{x}$, and for the action integrals of the form (4), also on the acceleration $\ddot{x}$. Namely, we first suppose the following structure of Δx and Δt:

$$\Delta x = \varepsilon F(t,x,\dot{x}), \quad \Delta t = \varepsilon f(t,x,\dot{x}), \quad 0<\varepsilon \ll 1, \tag{15}$$

$$\delta x = \varepsilon[F(t,x,\dot{x}) - \dot{x} f(t,x,\dot{x})] \neq 0. \tag{16}$$

We commence our further analysis from the so-called central Lagrangian equation which is of the form ([7], p. 259)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\delta x\right) = \delta L, \quad L = L(t,x,\dot{x}), \tag{17}$$

as one of the possible forms of the Euler-Lagrangian differential variational principle.

To derive the differential equation of motion, we note that

$$\delta L = \frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\delta \dot{x}, \tag{18}$$

and using this together with the commutative rules (11) we find immediately

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x}\right)\delta x = 0, \tag{19}$$

which is another possible form of the Lagrange-D'Alembert variational principle. Supposing that the variation δx is arbitrary (i.e., $\delta x \neq 0$), we arrive at the Euler-Lagrange differential equation (2). In doing this the internal structure of the variation (virtual displacement) is quite irrelevant.

To derive the action conservation law, we start from the central Lagrangian equation in the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\delta x\right) - \frac{\partial L}{\partial x}\delta x - \frac{\partial L}{\partial \dot{x}}\delta \dot{x} = 0, \tag{20}$$

and express the Lagrangian variations δx and $\delta \dot{x}$ in terms of generalized variations Δ [see (13) and (14) for $k=1$] to obtain

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}}(\Delta x-\dot{x}\,\Delta t)\right]-\frac{\partial L}{\partial x}(\Delta x-\dot{x}\,\Delta t)-\frac{\partial L}{\partial \dot{x}}(\Delta \dot{x}-\ddot{x}\,\Delta t)=0. \tag{21}$$

Expanding the last two terms on the left-hand side, regrouping, and adding and subtracting $\frac{\partial L}{\partial t}\Delta t$, we arrive at

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}}(\Delta x-\dot{x}\,\Delta t)\right]-\Delta L+\dot{L}\,\Delta t=0, \tag{22}$$

where

$$\Delta L=\frac{\partial L}{\partial t}\Delta t+\frac{\partial L}{\partial x}\Delta x+\frac{\partial L}{\partial \dot{x}}\Delta \dot{x} \tag{23}$$

and

$$\dot{L}=\frac{\partial L}{\partial t}+\frac{\partial L}{\partial x}\dot{x}+\frac{\partial L}{\partial \dot{x}}\ddot{x}. \tag{24}$$

Since $\dot{L}\,\Delta t=\frac{d}{dt}(L\Delta t)-L(\Delta t)^{\cdot}$, Eq. (22) can be rewritten as

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}}(\Delta x-\dot{x}\,\Delta t)+L\Delta t\right]-\Delta L-L(\Delta t)^{\cdot}=0 \tag{25}$$

or

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}}\Delta x+\left(L-\frac{\partial L}{\partial \dot{x}}\dot{x}\right)\Delta t\right]-\left[\frac{\partial L}{\partial t}\Delta t+\frac{\partial L}{\partial x}\Delta x+\frac{\partial L}{\partial \dot{x}}\Delta \dot{x}+L(\Delta t)^{\cdot}\right]=0. \tag{26}$$

Finally, by introducing the generators of infinitesimal transformations (15) and noting that $\Delta\dot{x}=\varepsilon(\dot{F}-\dot{x}\dot{f})$, the identity (26) becomes

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}}F+\left(L-\frac{\partial L}{\partial \dot{x}}\dot{x}\right)\dot{f}\right]-\left[\frac{\partial L}{\partial t}f+\frac{\partial L}{\partial x}F+\frac{\partial L}{\partial \dot{x}}\dot{F}+\left(L-\frac{\partial L}{\partial \dot{x}}\dot{x}\right)\dot{f}\right]=0. \tag{27}$$

By adding and subtracting the Lagrangian function $L=L(t,x,\dot{x})$, one has

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}}F+\left(L-\frac{\partial L}{\partial \dot{x}}\dot{x}\right)\dot{f}-\int L(t,x,\dot{x})dt\right]-\left[\frac{\partial L}{\partial x}F+\frac{\partial L}{\partial \dot{x}}\dot{F}+\left(L-\frac{\partial L}{\partial \dot{x}}\dot{x}\right)\dot{f}+\frac{\partial L}{\partial t}f-L(t,x,\dot{x})\right]=0. \tag{28}$$

From this equation we can derive the following result.

Theorem 1. For the generators of the infinitesimal transformations $F=F(t,x,\dot{x})$ and $f=f(t,x,\dot{x})$, which identically satisfy the equation

$$\frac{\partial L}{\partial x}F+\frac{\partial L}{\partial \dot{x}}\dot{F}+\left(L-\frac{\partial L}{\partial \dot{x}}\dot{x}\right)\dot{f}+\frac{\partial L}{\partial t}f-L(t,x,\dot{x})=0, \tag{29}$$

the dynamical system determined by the Lagrangian function $L=L(t,x,\dot{x})$ admits an action conservation law of the form

$$\frac{\partial L}{\partial \dot{x}}F+\left(L-\frac{\partial L}{\partial \dot{x}}\dot{x}\right)f-\int L(t,x,\dot{x})dt=\text{const}. \tag{30}$$

Note that the condition (29) is usually termed the Noether identity and from (19) it follows that the generators F and f are constrained by

$$\delta x=\varepsilon(F-\dot{x}f)\neq 0. \tag{31}$$

Naturally, if the dynamical system operates in a given interval of time $[t_0,t_1]$ the integral (30) reads

$$\left[\frac{\partial L}{\partial \dot{x}}F+\left(L-\frac{\partial L}{\partial \dot{x}}\dot{x}\right)f\right]_{t_0}^{t_1}-\int_{t_0}^{t_1}L(t,x,\dot{x})dt=0. \tag{32}$$

B. Case $L=L(t,x,\dot{x},\ddot{x})$

Let us now consider the transformation properties of the central Lagrangian equation for the case when Lagrangian function depends on the first- and second-order derivatives. For this case the central Lagrangian equation (the Lagrange-D'Alembert differential variational principle) reads as follows:

$$\delta L-\frac{d}{dt}\left[\left(\frac{\partial L}{\partial \dot{x}}-\frac{d}{dt}\frac{\partial L}{\partial \ddot{x}}\right)\delta x+\frac{\partial L}{\partial \ddot{x}}\delta\dot{x}\right]=0,\quad L=L(t,x,\dot{x},\ddot{x}). \tag{33}$$

It is easy to verify that this variational expression leads to the second form of the Lagrange-D'Alembert principle,

$$\left(\frac{\partial L}{\partial x}-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}+\frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{x}}\right)\delta x=0, \tag{34}$$

and considering δx as completely arbitrary, $\delta x\neq 0$, one arrives at the Euler-Lagrange equation (5).

Introducing again the generators of infinitesimal transformations,

$$\Delta x=\varepsilon F(t,x,\dot{x},\ddot{x}),\quad \Delta t=\varepsilon f(t,x,\dot{x},\ddot{x}),\quad 0<\varepsilon\ll 1, \tag{35}$$

$$\delta x=\varepsilon[F(t,x,\dot{x},\ddot{x})-\dot{x}f(t,x,\dot{x},\ddot{x})]\neq 0, \tag{36}$$

and substituting this into (33) [or (34)] we have, after easy but laborious transformations,

$$\varepsilon\left\langle\frac{\partial L}{\partial x}F+\frac{\partial L}{\partial \dot{x}}\dot{F}+\frac{\partial L}{\partial \ddot{x}}\ddot{F}-\left[L-\frac{\partial L}{\partial \ddot{x}}\ddot{x}+\left(\frac{d}{dt}\frac{\partial L}{\partial \ddot{x}}-\frac{\partial L}{\partial \dot{x}}\right)\dot{x}\right]\dot{f}-\frac{\partial L}{\partial t}f-L-\frac{d}{dt}\left\{\left(\frac{\partial L}{\partial \dot{x}}-\frac{d}{dt}\frac{\partial L}{\partial \ddot{x}}\right)F+\frac{\partial L}{\partial \ddot{x}}\dot{F}-\left[L-\frac{\partial L}{\partial \ddot{x}}\ddot{x}+\left(\frac{d}{dt}\frac{\partial L}{\partial \ddot{x}}-\frac{\partial L}{\partial \dot{x}}\right)\dot{x}\right]\dot{f}-\int L dt\right\}\right\rangle=0, \tag{37}$$

where we added and subtracted the Lagrangian function $\varepsilon L(t,x,\dot{x},\ddot{x})$.

From this expression we find the following result.

Theorem 2. For the generators $F=F(t,x,\dot{x},\ddot{x})$ and $f=f(t,x,\dot{x},\ddot{x})$, which identically satisfy the equation

$$\frac{\partial L}{\partial x}F+\frac{\partial L}{\partial \dot{x}}\dot{F}+\frac{\partial L}{\partial \ddot{x}}\ddot{F}-\left[L-\frac{\partial L}{\partial \ddot{x}}\ddot{x}+\left(\frac{d}{dt}\frac{\partial L}{\partial \ddot{x}}-\frac{\partial L}{\partial \dot{x}}\right)\dot{x}\right]\dot{f}-\frac{\partial L}{\partial t}f -L(t,x,\dot{x},\ddot{x})=0, \tag{38}$$

the dynamical system determined by the Lagrangian function $L=L(t,x,\dot{x},\ddot{x})$ admits an action conservation law of the form

$$\left(\frac{\partial L}{\partial \dot{x}}-\frac{d}{dt}\frac{\partial L}{\partial \ddot{x}}\right)F+\frac{\partial L}{\partial \ddot{x}}\dot{F}-\left[L-\frac{\partial L}{\partial \ddot{x}}\ddot{x}+\left(\frac{d}{dt}\frac{\partial L}{\partial \ddot{x}}-\frac{\partial L}{\partial \dot{x}}\right)\dot{x}\right]f -\int L(t,x,\dot{x},\ddot{x})dt=\text{const}. \tag{39}$$

The authors believe that the both theorems given in this section represent a specific form of Noether's theorem suitable for finding the action conservation laws of the Lagrangian dynamical systems given by $L(t,x,\dot{x})$ and $L(t,x,\dot{x},\ddot{x})$.

Note that traditionally Noether's theorem is based upon the invariant proper ties of the Hamilton action integral[1] $I=\int_{t_0}^{t_1}Ldt$ with respect to infinitesimal transformations of the generalized coordinate x and time t. However, here we have used the invariant properties of the central Lagrangian equation, since the invariant properties of the Lagrange-D'Alembert principle are rather *terra incognita* in analytical mechanics.

Ending this section we underline that in many practical situations finding the generators of infinitesimal transformations F and f for the given Lagrangian function L is not, as a rule, a difficult task. Thus, for example, for the harmonic oscillator discussed earlier, it is trivial to verify that for generators of the form $F=\frac{1}{2}x$ and $f=0$, the action integral (3) follows immediately from (30). Similarly, the problem given by (6) and (7) has the very same generators and the action integral (8) follows from (39).

III. IMPLEMENTATION

This section is devoted to a variety of boundary-value problems that can be analyzed using the action conservation laws from Theorems 1 and 2. Such an analysis usually consists of finding the unknowns that characterize a given boundary-value problem. These unknowns, depending on particular problem, can be the values of the function or its derivatives at the end points of the interval in a two-point boundary-value problem or the eigenvalue in a Sturm-Liouville class of problems. Whatever the unknown is, for a given second- or fourth-order boundary-value problem, we recommend the following procedure to be pursued in practice.

Find L. Verify that the given ordinary differential equation (ODE) is the Euler-Lagrange equation of the established Lagrangian L.

Find F and f. Use Noether's identity (29) or (38) to find a pair of generators F and f for the given Lagrangian L. Each pair of generators leads to a single conservation law.

[1]For the exhaustive exposition of the Noether theorem based upon the Hamilton principle, see [1].

Establish the conservation law. Use L, F, and f in (30) or (39).

Apply the conservation law to the end points. Use the boundary conditions specified for the problem, and this will yield algebraic relation among the parameters of the boundary value problem that are not specified.

Choose a trial solution. Like in other direct variational methods, construct a trial solution that satisfies the boundary conditions of the problem and substitute it in the algebraic relation obtained in previous step.

Solve. Solution of the algebraic equation yields the missing value.

This completes a direct procedure for the approximate, but very reliable, solution of boundary-value problems based on the utilization of conservation laws of the action type. It is equally applicable to linear, rheo-linear, and nonlinear boundary-value problems as will be illustrated on a number of examples.

However, an important remark should be made concerning the choice of a trial solution with or without adjustable parameters. A direct method described should always employ the fact that the action conservation law, when applied to the end points, contains the functional $I=\int_{t_0}^{t_1}Ldt$, which can be minimized separately with respect to adjustable parameters, and subsequently its optimal value is to be used in the evaluation of unknowns appearing in the algebraic relation arising from the conservation law. If the trial solution does not contain any adjustable parameter (like in Examples 1A and 1B), the unknown value can be obtained directly by applying the conservation law at the end points since the functional just becomes a number.

Next, we present several examples illustrating the usefulness of the action conservation laws.

A. Estimation of initial slope of the Thomas-Fermi problem

A classical test of various approximation methods is to estimate the behavior at the origin of the solution to the Thomas-Fermi differential equation corresponding to the electron distribution of a neutral atom. The boundary-value problem to be solved is

$$\ddot{x}=\sqrt{\frac{x^3}{t}},\quad \text{in } 0\leqslant t\leqslant\infty,\quad x(0)=1,\quad x(\infty)=0. \tag{40}$$

The Thomas-Fermi equation has the Lagrangian

$$L=\frac{1}{2}\dot{x}^2+\frac{2}{5}\sqrt{\frac{x^5}{t}}, \tag{41}$$

for which Noether's identity (29) becomes identically satisfied if the generators are chosen to be $F=\frac{3}{7}x$ and $f=-\frac{1}{7}t$. Consequently, the action conservation law (30) becomes

$$\frac{3}{7}x\dot{x}+\frac{1}{7}t\left(\frac{1}{2}\dot{x}^2-\frac{2}{7}\sqrt{\frac{x^5}{t}}\right)+\int Ldt=\text{const}, \tag{42}$$

and when this is applied to the end points and boundary conditions from (40) applied one arrives at

$$\dot{x}(0) = -\frac{7}{3}I(t,x,\dot{x}), \tag{43}$$

where the functional

$$I(t,x,\dot{x}) = \int_0^\infty \left(\frac{1}{2}\dot{x}^2 + \frac{2}{5}\sqrt{\frac{x^5}{t}}\right)dt. \tag{44}$$

Relations (42)–(44) have been presented by Vujanović and Jones ([1], p. 147), but here our aim is to estimate the slope of the solution at the origin $\dot{x}(0)$ from Eq. (43) by evaluating the value of the functional $\bar{I}=I(t,\bar{x},\dot{\bar{x}})$ at a reliable approximate solution $\bar{x}(t)$.

We will choose a three-parameter trial solution as proposed by Oulne [8] in the form

$$\bar{x}(t) = [1 + \alpha\sqrt{t} + \beta t \exp(-\gamma\sqrt{t})]^2 \exp(-2\alpha\sqrt{t}); \tag{45}$$

then, we will use the Ritz method to find the optimal values of the parameters α, β, and γ and finally evaluate $\bar{I}$ using these values.

Upon entering the trial function (45) into the functional and performing the integration, we obtain

$$\begin{aligned}\bar{I}(\alpha,\beta,\gamma) = {} & \frac{8\,776}{15\,625\alpha} + \frac{580\,608\beta^5}{9\,765\,625(\alpha+\gamma)^{11}} - \frac{96\alpha^5\beta}{(4\alpha+\gamma)^5} + \frac{8\beta}{(5\alpha+\gamma)^3} + \frac{3}{4}\alpha^4\beta\left[\frac{-5\beta}{(2\alpha+\gamma)^6} - \frac{48}{(4\alpha+\gamma)^4} + \frac{3\,840}{(5\alpha+\gamma)^7}\right] \\ & + \beta^2\left[\frac{3}{8(2\alpha+\gamma)^2} + \frac{192}{(5\alpha+2\gamma)^5}\right] + \alpha^3\beta\left[\frac{-3\beta}{4(2\alpha+\gamma)^5} - \frac{4}{(4\alpha+\gamma)^3} + \frac{1\,920}{(5\alpha+\gamma)^6} + \frac{40\,320\beta}{(5\alpha+2\gamma)^8} - \frac{320\beta^2}{(4\alpha+3\gamma)^7}\right] \\ & + \frac{32}{3}\beta^3\left[(4\alpha+3\gamma)^{-4} + \frac{540}{(5\alpha+3\gamma)^7}\right] + \beta^4\left[\frac{75}{4\,096(\alpha+\gamma)^6} + \frac{161\,280}{(5\alpha+4\gamma)^9}\right] + \alpha^2\Bigg\{\frac{19}{128} + \frac{576\beta}{(5\alpha+\gamma)^5} + \beta^2\Bigg[\frac{3}{4(2\alpha+\gamma)^4} \\ & + \frac{17\,280}{(5\alpha+2\gamma)^7}\Bigg] + \frac{80}{3}\beta^3\left[(4\alpha+3\gamma)^{-6} + \frac{12\,096}{(5\alpha+3\gamma)^9}\right]\Bigg\} + \alpha\Bigg\{\frac{96\beta}{(5\alpha+\gamma)^4} + \frac{1\,451\,520\beta^4}{(5\alpha+4\gamma)^{10}} + \beta^2\Bigg[\frac{3}{4(2\alpha+\gamma)^3} \\ & + \frac{2\,880}{(5\alpha+2\gamma)^6}\Bigg] + \frac{128}{3}\beta^3\Bigg] \; (4\alpha+3\gamma)^{-5} + \frac{1\,890}{(5\alpha+3\gamma)^8}\Bigg]\Bigg\}.\end{aligned} \tag{46}$$

Three algebraic equations

$$\partial_\alpha\bar{I}(\alpha,\beta,\gamma) = 0, \quad \partial_\beta\bar{I}(\alpha,\beta,\gamma) = 0, \quad \partial_\gamma\bar{I}(\alpha,\beta,\gamma) = 0 \tag{47}$$

are formed and solved numerically to yield

$$\alpha = 0.706\,617, \quad \beta = -0.557\,466, \quad \gamma = 0.368\,481, \tag{48}$$

and the approximate solution to the problem is thus completed.

The functional at the approximate solution is

$$\bar{I}(0.706\,617,-0.557\,466,0.368\,481) = 0.680\,609\,627\,773, \tag{49}$$

which compares well with the exact value ([9], p. 79): $I = 0.6806$. The initial slope of the solution as obtained by using (49) in (43) is

$$\dot{x}(0) = -1.588\,089\,131\,47. \tag{50}$$

This is in perfect agreement with the numerical result −1.588 071 02 of Lee and Wu [10] and Kobayashi *et al.* [11]. Note that our solution is better then the one by Oulne [8] who proposed the trial solution that we used. Actually it is better than any known solution. This is due to the fact that the minimal property of the Hamilton action integral has been used together with the evaluation the action conservation law.

B. Column stability problem of Atanacković and Simić

In studying the shape of the Pflüger column of greatest efficiency Atanacković and Simić [12] and Atanacković [13] derived second-order nonlinear ODEs describing the moment $x(\xi)$ in the first buckling mode of the optimally shaped column. Both these equations can be unified by the two-point boundary-value problem

$$\ddot{x} = \left(\frac{\xi}{x}\right)^{1/3} \quad \text{in } a \leqslant \xi \leqslant b, \quad x(a) = x(b) = 0, \tag{51}$$

where

$$a = -(\lambda_1 + \lambda_2)\lambda_1^{-2/3} \quad \text{and } b = -\lambda_2\lambda_1^{-2/3}. \tag{52}$$

λ_1 and λ_2 are the critical distributed load of the Pflüger column of constant cross section and the applied dimensionless axial force, respectively. These two quantities are the first non-negative roots of the transcendental equation

$$\frac{\mathrm{Ai}(-(\lambda_1+\lambda_2)\lambda_1^{-2/3})}{\mathrm{Bi}(-(\lambda_1+\lambda_2)\lambda_1^{-2/3})} = \frac{\mathrm{Ai}(-\lambda_2\lambda_1^{-2/3})}{\mathrm{Bi}(-\lambda_2\lambda_1^{-2/3})} \tag{53}$$

involving the Airy functions.

The results of practical interest for the problem described above are the slopes $\dot{x}(a)$ and $\dot{x}(b)$ at the end points being the measures of the reaction forces acting at the supports of the column. The nonlinear boundary-value problem (51) and (52) can be solved numerically; however, our goal is to provide estimates of these slopes by using the conservation law of the action integral type which we construct next.

First, note that the nonlinear ODE (51) is the Euler-Lagrange equation for the Lagrangian

$$L = \frac{1}{2}\dot{x}^2 + \frac{3}{2}(\xi x^2)^{1/3}. \tag{54}$$

By using the general Noether identity one can establish that the generators of the form $F=\frac{7}{10}x(\xi)$ and $f=\frac{2}{5}\xi$ make the following quantity conserved:

$$2\xi\dot{x}^2 - 7x\dot{x} - 6\xi^{4/3}x^{2/3} + 10\int L dt = \text{const}. \tag{55}$$

When this differential invariant is applied to the boundary-value problem (51) and (52) it becomes obvious that the slopes at the end points are related:

$$a\dot{x}^2(a) - b\dot{x}^2(b) = 5\int_a^b L dt. \tag{56}$$

Example A. Consider the problem of Atanacković and Simić [12,13] in original variables. The independent variable ξ in (51) is related to the original variables by $\xi=-[\lambda_1(1-t)+\lambda_2]\lambda_1^{-2/3}$, where $0\leq t\leq 1$ is dimensionless coordinate along the column length, while the dependent variable $x=\sqrt{2}m$, where $m=m(t)$ is dimensionless bending moment. Thus the problem reads

$$\ddot{m} + \left[\frac{\bar{\lambda}_1(1-t)+\bar{\lambda}_2}{m}\right]^{1/3} = 0 \quad \text{in } 0\leq t\leq 1,$$

$$m(0) = m(1) = 0, \tag{57}$$

where $\bar{\lambda}_i=\lambda_i/4$, $(i=1,2)$.

First of all note that the Lagrangian of the problem is

$$L = \frac{1}{2}\dot{m}^2 - \frac{3}{2}\{[\bar{\lambda}_1(1-t)+\bar{\lambda}_2]m^2\}^{1/3} \tag{58}$$

and that the general Noether identity yields the generators $F=\frac{7}{10}m(t)$ and $f=-\frac{2}{5}[\bar{\lambda}_1(1-t)+\bar{\lambda}_2]\bar{\lambda}_1^{-1}$, which make the following quantity conserved:

$$-7m\dot{m} - 4[\bar{\lambda}_1(1-t)+\bar{\lambda}_2]\bar{\lambda}_1^{-1}\left\{\frac{1}{2}\dot{m}^2 + \frac{3}{2}[\bar{\lambda}_1(1-t) + \bar{\lambda}_2]^{1/3}m^{2/3}\right\} + 10\int L dt = \text{const}. \tag{59}$$

Applying this to the boundary-value problem (57) we obtain the relation among the slopes at the end points:

$$\bar{\lambda}_2\dot{m}^2(1) - (\bar{\lambda}_1+\bar{\lambda}_2)\dot{m}^2(0) = 5\bar{\lambda}_1\int_0^1 L dt. \tag{60}$$

The results of Atanacković and Simić [12] correspond to the case $\lambda_2=0$ and $\lambda_1=18.956\ 265\ 591\ 4$—i.e., the Pflüger rod without concentrated axial load. In this example we want to estimate the initial slope of the bending moment $\dot{m}(0)$. For such case we have, from (60) with $\bar{\lambda}_2=0$,

$$\dot{m}^2(0) = -5I, \tag{61}$$

where the functional

$$I = \int_0^1 L(t,m,\dot{m})dt \tag{62}$$

with the Lagrangian $L(t,m,\dot{m})=\frac{1}{2}\dot{m}^2-\frac{3}{2}\{[\bar{\lambda}_1(1-t)]m^2\}^{1/3}$ according to (58) for $\bar{\lambda}_2=0$.

Trial solutions $\bar{m}_n(t)=c_0t(1-t)(1+\Sigma_{i=1}^n c_it^i)$ with $n=0,1,2$ have been used and the results presented in Table I are obtained from

$$\dot{m}(0) = \sqrt{-5\min\bar{I}}, \tag{63}$$

where $\bar{I}=\int_0^1 L(t,\bar{m},\dot{\bar{m}})dt$ is minimized with respect to adjustable constants $(\partial_{c_i}\bar{I}=0)$. Note in Table I how the estimates of the initial slope $\dot{\bar{m}}(0)$ directly from the trial solution are poor when compared those obtained from (63).

TABLE I. Prediction of initial slope for the optimized Pflüger's rod.

n	$\min\bar{I}$	c_n	$\dot{\bar{m}}(0)$	$\dot{m}(0)$
0	−0.402 421	c_0=1.098 755	1.098 755	1.418 487
1	−0.404 894	c_0=1.244 687	1.244 687	1.422 839
		c_1=−0.238 754		
2	−0.405 165	c_0=1.303 566	1.303 566	1.423 315
		c_1=−0.456 295		
		c_2=0.229 271		
Numerical integration				1.422 185

C. Arbitrary unforced rheo-linear problems

If the governing equation of a dynamical system is of the rheo-linear form

$$f_2(t)\ddot{x} + f_1(t)\dot{x} + f_0(t)x = 0, \quad \text{in } t_0\leq t\leq t_f, \tag{64}$$

with arbitrary functions $f_2(t)\neq 0$, $f_1(t)$, and $f_0(t)$, a conservation law of the form

$$C_1 = x(t)\dot{x}(t)\exp\left(\int\frac{f_1}{f_2}dt\right) - \int\left(\dot{x}^2 - \frac{f_0}{f_2}x^2\right)\exp\left(\int\frac{f_1}{f_2}dt\right)dt \tag{65}$$

holds for generating functions taken as $F=\frac{x}{2}$ and $f=0$. Consequently, at the exact solution of this kind of problems the following relation is valid:

$$x(t)\dot{x}(t)\exp\left(\int\frac{f_1}{f_2}dt\right)\Bigg|_{t_0}^{t_f} = \int_{t_0}^{t_f}\left(\dot{x}^2 - \frac{f_0}{f_2}x^2\right)\exp\left(\int\frac{f_1}{f_2}dt\right)dt. \tag{66}$$

Note that in the case of the undumped ($f_1=0$) equation (64), relation (66) reduces to

$$x(t)\dot{x}(t)|_{t_0}^{t_f} = \int_{t_0}^{t_f}\left(\dot{x}^2 - \frac{f_0}{f_2}x^2\right)dt. \quad (67)$$

Example B. Suppose that we want to estimate the initial slope $\dot{x}(0)$ for the boundary-value problem described by the Weber equation

$$\ddot{x} + \left(\frac{1}{4} - t^2\right)x = 0, \quad \text{in } 0 \leqslant t \leqslant \frac{1}{2}, \quad x(0)=1, \quad \dot{x}\left(\frac{1}{2}\right) = 0. \quad (68)$$

An exact solution of this problem can be expressed in terms of the Hermite polynomials and the Kummer confluent hypergeometric function, and the exact value that we are looking for is

$$\dot{x}(0) = \frac{2\Gamma\left(\frac{19}{16}\right)\left[4\,{}_1F_1\left(\frac{3}{16};\frac{1}{2};\frac{1}{4}\right) - 3\,{}_1F_1\left(\frac{19}{16};\frac{3}{2};\frac{1}{4}\right)\right]}{3\Gamma\left(\frac{3}{16}\right){}_1F_1\left(\frac{11}{16};\frac{1}{2};\frac{1}{4}\right) - 4\Gamma\left(\frac{19}{16}\right){}_1F_1\left(\frac{11}{16};\frac{3}{2};\frac{1}{4}\right)} = 0.084\,161\,7.$$

This result can be easily estimated from the relation

$$\dot{x}(0) = -\int_0^{1/2}\left[\dot{x}^2 - \left(\frac{1}{4} - t^2\right)x^2\right]dt, \quad (69)$$

which follows for this problem from (67). If we choose a trial function $\bar{x}(t)=1+c(t-1)t$ satisfying the boundary conditions in (68) and evaluate the functional in (69), we obtain $J(c)=-\int_0^{1/2}[\dot{\bar{x}}^2-(\frac{1}{4}-t^2)\bar{x}^2]dt$ $=[-1120+c(308+2213c)]/13\,440$, which can be minimized $[\partial_c J(c)=0]$ to yield an optimal value for the adjustable constant $c=-\frac{154}{2213}$. Then $J(c)=-\frac{29\,789}{354\,080}=-0.084\,130\,7=-\dot{x}(0)$. This value underpredicts the exact result for only 0.037%.

D. Derivation of the Rayleigh quotients from conservation laws

The structure of the Lagrangian for the formally self-adjoint homogeneous ODE of the 2*k*th order,

$$\sum_{i=0}^{k}(-1)^i\frac{d^i}{dt^i}\left(f_i(t)\frac{d^ix}{dt^i}\right) = 0, \quad (70)$$

is

$$L = \frac{1}{2}\sum_{i=0}^{k} f_i(t)\left(\frac{d^ix}{dt^i}\right)^2, \quad (71)$$

and for each such a problem it can be proved that the corresponding Noether identity will be satisfied by generators of the form $F=\frac{1}{2}x$ and $f=0$. This means that the action integral conservation law exists for each such a problem. Here we are interested in the second-order ($k=1$) and fourth-order ($k=2$) boundary-value problems of the Sturm-Liouville type—i.e., the self-adjoint eigenvalue problems.

For $k=1$ consider the Sturm-Liouville equation of the form

$$-(f_1(t)\dot{x})^{\cdot} + [f_0(t) - \lambda g_0(t)]x = 0 \quad (72)$$

in $t\in[a,b]$ with the Lagrangian

$$L(t,x,\dot{x}) = \frac{1}{2}f_1(t)\dot{x}^2 + \frac{1}{2}[f_0(t) - \lambda g_0(t)]x^2, \quad (73)$$

which leads, with $F=\frac{1}{2}x$ and $f=0$ used in (30), to the action conservation law of the form

$$\frac{1}{2}f_1 x\dot{x} - \int L(t,x,\dot{x})dt = \text{const}. \quad (74)$$

When the Lagrangian (73) is introduced and (74) applied to the end points the following algebraic relation is obtained:

$$[f_1 x\dot{x}]_a^b - \int_a^b (f_1\dot{x}^2 + f_0 x^2)dt + \lambda\int_a^b g_0 x^2 dt = 0, \quad (75)$$

where from the well-known Rayleigh quotient follows

$$\lambda = \frac{\int_a^b (f_1\dot{x}^2 + f_0x^2)dt - [f_1x\dot{x}]_a^b}{\int_a^b g_0x^2dt}. \quad (76)$$

Similarly, the Rayleigh quotient of the form

$$\lambda = \frac{\int_a^b (f_2\ddot{x}^2 + f_1\dot{x}^2 + f_0x^2)dt + [f_2\dot{x}\ddot{x} + (\dot{f}_2x - f_2\dot{x})\ddot{x} - f_1x\dot{x}]_a^b}{\int_a^b (g_0x^2 + g_1\dot{x}^2)dt - [g_1x\dot{x}]_a^b} \quad (77)$$

can be derived for the eigenvalue problems of the fourth order,

$$(f_2(t)\ddot{x})^{\cdot\cdot} - \{[f_1(t) - \lambda g_1(t)]\dot{x}\}^{\cdot} + [f_0(t) - \lambda g_0(t)]x = 0, \quad (78)$$

whose Lagrangian is

$$L(t,x,\dot{x},\ddot{x}) = \frac{1}{2}f_2(t)\ddot{x}^2 + \frac{1}{2}[f_1(t) - \lambda g_1(t)]\dot{x}^2 + \frac{1}{2}[f_0(t) - \lambda g_0(t)]x^2, \quad (79)$$

and the action conservation law is (since again the generators are $F=\frac{1}{2}x$ and $f=0$)

$$\frac{1}{2}[x(f_2\ddot{x})^{\cdot} - f_2\dot{x}\ddot{x} - (f_1 - \lambda g_1)x\dot{x}] + \int L(t,x,\dot{x},\ddot{x})dt = \text{const}. \quad (80)$$

It is a common practice in evaluating the first eigenvalue from the Rayleigh quotients for a particular boundary-value problem that a minimization of the quotient is performed over a class of trial functions satisfying the boundary condi-

TABLE II. The first eigenvalue for parallel plates.

Trial solution	λ
$1-t^2$	1.707 83
$\cos(\frac{\pi}{2}t)$	1.684 74
$1-(1+b)t^2+bt^4$	1.682 15
Exact eigenvalue:	1.681 60

TABLE III. The first eigenvalue for cylinder.

Trial solution	λ
$1-t^2$	2.828 43
$\cos(\frac{\pi}{2}t)$	2.756 74
$1-(1+b)t^2+bt^4$	2.709 27
Exact eigenvalue:	2.704 36

tions. It is also true in our case, and we present three illustrative examples in order to show accurate estimations of the eigenvalues by minimizing not just the quotient, but the functional as well.

Example C: Graetz problem for parallel plates. Here we study the estimation of the first eigenvalue of the Graetz problem for parallel plates. The boundary-value problem to be solved is

$$\ddot{x}+\lambda^2(1-t^2)x=0, \quad \text{in } 0\leqslant t\leqslant 1, \quad \dot{x}(0)=0, \quad x(1)=0. \tag{81}$$

At the exact solution of this problem the following constant holds:

$$x(t)\dot{x}(t)\big|_{t=0}^{t=1}=\int_0^1[\dot{x}^2-\lambda^2(1-t^2)x^2]dt, \tag{82}$$

since the functional for the differential equation (81) is of the form

$$I(t,x,\dot{x})=\frac{1}{2}\int_0^1[\dot{x}^2-\lambda^2(1-t^2)x^2]dt. \tag{83}$$

As the boundary conditions for this problem, make the right-hand term of Eq. (82) vanish; we are faced with the situation where the action integral is identically equal to zero:

$$I(t,x,\dot{x})=0. \tag{84}$$

This fact makes a possibility to evaluate an eigenvalue of the problem from the ratio

$$\lambda^2=\frac{\int_0^1\dot{x}^2dt}{\int_0^1(1-t^2)x^2dt}. \tag{85}$$

This can be done by using in Eq. (85) instead of $x(t)$ various trial functions $\bar{x}(t)$ satisfying the boundary conditions to the problem. For example, one might chose a polynomial of the form $\bar{x}(t)=1-t^2$ or a circular function like $\bar{x}(t)=\cos(\frac{\pi}{2}t)$. Table II summarizes the results for various trial functions.

The trial function $\bar{x}(t)=1-(1+b)t^2+bt^4$ having one additional parameter b deserves additional comment. Besides the condition that the functional has to vanish in this problem, we have still a possibility to minimize, say, by the Ritz method, the functional. In other words, if we have an additional unknown, except λ, say, b, we may use two equations

$$\bar{I}\equiv I(t,\bar{x},\dot{\bar{x}})=0 \quad \text{and} \quad \partial_b\bar{I}=0 \tag{86}$$

to determine both λ and b. The value of the eigenvalue λ given if Table II, for trial function containing b, follows the form of Eq. (85) as

$$\lambda=\frac{\sqrt{33}}{2}\sqrt{\frac{35-14b+11b^2}{99-22b+3b^2}} \tag{87}$$

for arbitrary b, but an optimal value of b appears to be $b=\frac{1}{25}(123-2\sqrt{3\,301})=0.323\,654$ as obtained by minimizing the functional at the approximate solution,[2] and this leads to the value of λ presented in Table II.

The exact eigenvalue given in Table II for this problem is the first root of the Kummer confluent hypergeometric function ${}_1F_1(\frac{1-\lambda}{4},\frac{1}{2},\lambda)=0$ (see [14]). Thus, our optimized estimation of the first eigenvalue underpredicts the exact value absolutely for -0.023 or for 0.033%.

Example D: Graetz problem for a cylinder. The boundary-value problem to be solved is

$$t\ddot{x}+\dot{x}+\lambda^2(1-t^2)tx=0, \quad \text{in } 0\leqslant t\leqslant 1,$$

$$\dot{x}(0)=0, \quad x(1)=0. \tag{88}$$

The functional in this case is

$$I(t,x,\dot{x})=\frac{1}{2}\int_0^1[\dot{x}^2-\lambda^2(1-t^2)x^2]t\,dt. \tag{89}$$

Using the same type of trial functions as above, but this time evaluating λ from

$$\lambda^2=\frac{\int_0^1 t\dot{x}^2dt}{\int_0^1 t(1-t^2)x^2dt}, \tag{90}$$

we obtain results as presented in Table III.

The exact eigenvalue given in Table III for this problem is the first root of the Laguerre polynomial $L_{(\lambda-2)/4}(\lambda)=0$. The adjustable trial function $\bar{x}(t)=1-(1+b)t^2+bt^4$ leads to

[2]It appears that one might minimize λ, given by Eq. (85), with respect to the parameter b.

$$\lambda = \frac{2\sqrt{10}}{2}\sqrt{\frac{3-2b+b^2}{15-6b+b^2}}, \tag{91}$$

and after minimizing λ, an optimal value of b is found to be $b=3-\sqrt{6}$, leading to the result that overpredicts the exact value of the first eigenvalue for just 0.18%.

Change in boundary conditions. Suppose we have to evaluate the first eigenvalue for the problem like in (88), but for boundary conditions corresponding to the constant heat flux at the tube wall:

$$t\ddot{x}+\dot{x}+\lambda^2(1-t^2)tx=0, \quad \text{in } 0\leqslant t\leqslant 1, \quad \dot{x}(0)=0,$$

$$\dot{x}(1)=1. \tag{92}$$

In this case we have to evaluate

$$\lambda^2 = \frac{-x(1)+\int_0^1 t\dot{x}^2 dt}{\int_0^1 t(1-t^2)x^2 dt}, \tag{93}$$

at some trial function satisfying boundary conditions in (92). An appropriate trial function containing adjustable constant c is $\bar{x}(t)=1+ct^2+\frac{1}{3}(1-2c)t^3$ and leads to

$$\lambda = \frac{6\sqrt{7}\sqrt{-35+2(-3+c)c}}{\sqrt{2199+c(760+79c)}}. \tag{94}$$

The optimal value, obtained from $\partial_c\lambda=0$, is $c=(-7163-\sqrt{24\,577\,005})/1994$ so that finally

$$\lambda = 3\sqrt{\frac{14(3913+\sqrt{24577005})}{29321}} = 6.17405. \tag{95}$$

This overpredicts the exact value[3] $\lambda=6.125\,87$ for just 0.78%.

Example E. Consider another problem of lateral displacement of an inextensible elastic rod pinned at both ends and loaded by two axial concentrated forces, like in Example 1B, but this time governed by the following fourth-order boundary-value problem [2], p. 172:

$$((3-t^2)\ddot{x})^{\cdot\cdot}+\lambda\ddot{x}+60x=0, \quad \text{with } x(\pm 1)=\ddot{x}(\pm 1)=0. \tag{96}$$

By identifying the rheonoms of this problem $f_i(t)$, $i=0,1,2$, and $g_i(t)$, $i=0,1$, in (78) as $f_2=3-t^2$, $f_1=g_0=0$, $f_0=60$, and $g_1=1$, we have the Lagrangian of the form (79):

$$L(t,x,\dot{x},\ddot{x}) = \frac{1}{2}(3-t^2)\ddot{x}^2 - \frac{\lambda}{2}\dot{x}^2+30x^2. \tag{97}$$

The action conservation law of the form (80) is

TABLE IV. The first eigenvalue of (96) with various trial solutions.

Trial solution	λ
$(1-t^2)\ (3433-599t^2-53t^4+3t^6)$	31.3607
$(1-t^2)\ (39-6t^2-t^4)$	31.3548
$(1-t^2)\ [9\alpha+9(\alpha+5)t^2-(6\alpha+25)t^4]$	31.3529
Estimate [2], pp. 215–217:	$31.350\,96<\lambda<31.354\,85$

$$\frac{1}{2}[(3-t^2)(x\dddot{x}-\dot{x}\ddot{x})-2t\dot{x}\ddot{x}+\lambda x\dot{x}]+\int L(t,x,\dot{x},\ddot{x})dt=\text{const}. \tag{98}$$

To find the smallest eigenvalue λ for this problem, we apply the conservation law (98) to the endpoints and conclude, due to the nature of boundary conditions) that the functional

$$I=\int_{-1}^{+1} L(t,x,\dot{x},\ddot{x})dt=0, \tag{99}$$

which is equivalent to the following Rayleigh quotient of the type (77):

$$\lambda = \frac{\int_{-1}^{+1}[(3-t^2)\ddot{x}^2+60x^2]dt}{\int_{-1}^{+1}\dot{x}^2 dt}. \tag{100}$$

We used three trial functions $\bar{x}(t)$ satisfying the boundary conditions to the problem and evaluated the eigenvalue λ as presented in Table IV. The first two trial polynomials are without an adjustable parameter and λ is calculated directly from (100). For the third polynomial, containing adjustable parameter α, two equations are used: zero value of the functional (99) and $\partial_\alpha I=0$ evaluated at trial solution, to determine both $\lambda=\frac{87\,285-\sqrt{840\,998\,895}}{1\,859}=31.3529$ and $\alpha=\frac{-224\,485-6\sqrt{840\,998\,895}}{90\,249}=-4.41539$.

E. Conservation laws for some fourth-order ODEs

Recently Everitt *et al.* [15] reported the representation of the solutions of four fourth-order-type (named as Bessel-Laguerre-Legendre-Jacobi) ordinary differential equations. All these equations are written in formally self-adjoint form, and we are able immediately to write down action conservation laws for each of them. Namely, by simple identification of the rheonoms $f_i(t)$, $i=0,1,2$, and $g_i(t)$, $i-0,1$, in (78) we can establish conservation laws of the form (80) with the Lagrangians of the form (79).

Example F. Bessel-type fourth-order equation. For the differential equation of the form

$$(t\ddot{x})^{\cdot\cdot}-[(9t^{-1}+8tM^{-1})\dot{x}]^{\cdot}-\lambda^2 t(\lambda^2+8M^{-1})x=0, \tag{101}$$

for all $t\in(0,\infty)$, where $M\in(0,\infty)$ is a positive parameter and $\lambda\in\mathbb{C}$, we have $f_2=t$, $f_1=9t^{-1}+8tM^{-1}$, $f_0=g_1=0$, and $g_0=\lambda^2 t(\lambda^2+8M^{-1})$ and the Lagrangian

[3]Note that the exact value is the first root of the transcendental equation

$$\lambda e^{-\lambda/2}[L_{(\lambda-2)/4}(\lambda)+2L^1_{(\lambda-6)/4}(\lambda)]+1=0.$$

$$L(t,x,\dot{x},\ddot{x}) = \frac{1}{2}t\ddot{x}^2 + \frac{1}{2}(9t^{-1} + 8tM^{-1})\dot{x}^2 - \frac{1}{2}\lambda^2 t(\lambda^2 + 8M^{-1})x^2, \tag{102}$$

so that the conservation law is

$$\frac{1}{2}[t(x\dddot{x} - \dot{x}\ddot{x}) + x\ddot{x} - (9t^{-1} + 8tM^{-1})x\dot{x}] + \int L(t,x,\dot{x},\ddot{x})dt = \text{const.} \tag{103}$$

Example G. Laguerre-type fourth-order equation. For the differential equation of the form

$$(t^2e^{-t}\ddot{x})^{\cdot\cdot} - [2e^{-t}(At + t + 1)\dot{x}]^{\cdot} - \lambda e^{-t}x = 0, \tag{104}$$

for all $t \in (0,\infty)$, where $A \in (0,\infty)$ is a positive parameter and $\lambda \in \mathbb{C}$, we have $f_2=t^2e^{-t}$, $f_1=2e^{-t}(At+t+1)$, $f_0=g_1=0$, and $g_0=e^{-t}$ and the Lagrangian

$$L(t,x,\dot{x},\ddot{x}) = \frac{1}{2}e^{-t}[t^2\ddot{x}^2 + 2(A + t + 1)\dot{x}^2 - \lambda x^2], \tag{105}$$

so that the conservation law is

$$\frac{1}{2}e^{-t}[t^2(x\dddot{x} - \dot{x}\ddot{x}) - t(t-2)x\ddot{x} - 2(At + t + 1)x\dot{x}] \tag{106}$$

$$+ \int L(t,x,\dot{x},\ddot{x})dt = \text{const.} \tag{107}$$

Example H. Legendre-type fourth-order equation. For the differential equation of the form

$$((1-t^2)^2\ddot{x})^{\cdot\cdot} - \{[8 + 4A(1-t^2)]\dot{x}\}^{\cdot} - \lambda x = 0, \tag{108}$$

for all $t \in (-1,+1)$, where $A \in (0,\infty)$ is a positive parameter and $\lambda \in \mathbb{C}$, we have $f_2=(1-t^2)^2$, $f_1=8+4A(1-t^2)$, $f_0=g_1=0$, and $g_0=1$ and the Lagrangian

$$L(t,x,\dot{x},\ddot{x}) = \frac{1}{2}\{(1-t^2)^2\ddot{x}^2 + [8 + 4A(1-t^2)]\dot{x}^2 - \lambda x^2\}, \tag{109}$$

so that the conservation law is

$$\frac{1}{2}\{(1-t^2)^2(x\dddot{x} - \dot{x}\ddot{x}) - 4t(1-t^2)x\ddot{x} - [8 + 4A(1-t^2)]x\dot{x}\} + \int L(t,x,\dot{x},\ddot{x})dt = \text{const.} \tag{110}$$

Example I. Jacobi-type fourth-order equation. For the differential equation of the form

$$((1-t)^{\alpha+2}(1+t)^2\ddot{x})^{\cdot\cdot} - \{2(1-t)^{1+\alpha}[\alpha + 3 + t(\alpha+1) + 2^{\alpha+1}A(1+t)]\dot{x}\}^{\cdot} - \lambda(1-t)^{\alpha}x = 0, \tag{111}$$

for all $t \in (-1,+1)$, where $A \in (0,\infty)$, $\alpha \in (-1,\infty)$, and $\lambda \in \mathbb{C}$, we have $f_2=(1-t)^{\alpha+2}(1+t)^2$, $f_1=2(1-t)^{1+\alpha}[\alpha+3+t(\alpha+1)+2^{\alpha+1}A(1+t)]$, $f_0=g_1=0$, and $g_0=(1-t)^{\alpha}$ and the Lagrangian

$$L(t,x,\dot{x},\ddot{x}) = \frac{1}{2}(1-t)^{\alpha+2}(1+t)^2\ddot{x}^2 + (1-t)^{1+\alpha}[\alpha + 3 + t(\alpha+1) + 2^{\alpha+1}A(1+t)]\dot{x}^2 - \frac{1}{2}\lambda(1-t)^{\alpha}x^2, \tag{112}$$

so that the conservation law is

$$\frac{1}{2}(1-t)^{\alpha}\{(1-t^2)^2(x\dddot{x} - \dot{x}\ddot{x}) - (1-t^2)[\alpha + (\alpha+4)t]x\ddot{x} - 2(1-t)[\alpha + 3 + t(\alpha+1) + 2^{\alpha+1}A(1+t)]x\dot{x}\} + \int L(t,x,\dot{x},\ddot{x})dt = \text{const.} \tag{113}$$

Once the boundary conditions are specified for any of the ODEs in examples III F–III I further study of the boundary-value problems becomes available via their conservation laws.

IV. CONCLUDING REMARKS

Action conservation laws are derived from the invariant properties of the Lagrange-D'Alembert differential variational principle. Their general form as in Eq. (1) for the second-order ODEs and in Eq. (4) for the fourth-order ODEs consists of action integrals and the functions depending, except for the independent and dependent variables, on the derivatives up to the order by one less than the original problem. So they are the first integrals of the Euler-Lagrange equations. The structure of these conservation laws is closely related to the natural boundary conditions of the variational problem, and this fact makes it possible to develop a reliable approximate method for direct study of various boundary-value problems in physics and mechanics once the conservation law is applied to the end points.

ACKNOWLEDGMENTS

The authors thank Professor T. M. Atanacković for the discussions and helpful suggestions.

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